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SU(n) anomaly generating functionals: a toolkit for model builders

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Abstract. We give the functionals that yield the leading anomaly coefficients $A_{(k)}(\lambda)$ for SU(n) gauge groups subject to $2 \leqslant k \leqslant n$ and representations λ whose Young diagrams have less than seven boxes. This covers all widely used representations. Further we give an example indicating how to calculate beyond the stable range where $k > n$.

1. Introduction

The study of global properties of field theories has been much developed in the past decade (for a review see Alvarez-Gaumé 1985). In particular the massless spectrum of a theory and the not unrelated question of anomalies in the regularisation of the quantum theory have been widely investigated. The requirement of the absence of anomalies can provide a powerful constraint on the matter multiplets appearing in a gauge theory even if the gauge group is broken (see 't Hooft 1980 and Slansky 1981). A common approximation is to assume a pattern of symmetry breaking whereby only the massless modes of some initial theory survive to low energies at which stage some dynamical symmetry breaking gives them masses. The (initially) massless spectrum is that which is observed in present day experiments. Both of these properties become purely topological questions in the framework of the Euclidean path integral.

The focus of this paper will be to provide results useful for analysing the anomaly structure of a gauge theory. Suppose we are interested in a $2d$-dimensional gauge theory coupled to chiral fermions in a representation $V(\lambda)$ of the gauge group. In the evaluation of the chiral fermion determinants one encounters an expression $\mathrm{Tr}_{V(\lambda)} F^{d+1}$, the trace of the gauge field strength in the representation $V(\lambda)$. This trace is related to traces and their powers of a fundamental representation $V(\lambda_f)$ by

$$\mathrm{Tr}_{V(\lambda)} F^{d+1} = A_{(d+1)}(\lambda)\, \mathrm{Tr}_{V(\lambda_f)} F^{d+1} + \sum_{\pi}{}' A_{\pi}(\lambda) (\mathrm{Tr}_{V(\lambda_f)} F^{k_1})^{\pi_1} \ldots (\mathrm{Tr}_{V(\lambda_f)} F^{k_s})^{\pi_s}. \quad (1.1)$$

The sum in this formula is over partitions $\pi = (k_1^{\pi_1}, \ldots, k_s^{\pi_s})$ of $d+1$ with $s \geqslant 2$, the prime denoting we have separated off the $s=1$ term $\pi = (d+1)$ for special attention. This term is known as the 'leading anomaly coefficient' and is that term relevant for calculating on a (Euclidean) spacetime with topology of the sphere, S^{2d}. Summing $A_{(d+1)}(\lambda_i)$ over the fermion representations of the theory then gives the overall anomaly for this topology. Clearly other topologies are possible (and may be calculated for) but S^{2d} is the simplest compactification of Wick rotated Minkowski space and so most often treated. In the following we let $k = d+1$.

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The aim of the present paper is to present a series of tables which give the leading anomaly coefficients $A_{(k)}(\lambda)$ for an SU(n) gauge theory. Here λ_f is the n-dimensional representation and these coefficients are useful for discussing global (Witten 1982, Elitzur and Nair 1984, Braden 1988a) as well as perturbative anomalies. To facilitate use we will specify a representation by its corresponding Young diagram. For each such diagram, $A_{(k)}(\lambda)$ is then a polynomial in k and n for which the reader can substitute as desired, subject only to $2 \leqslant k \leqslant n$. For $k=1$ these coefficients are zero because of the tracelessness of su(n) and so throughout we will only consider $k \geqslant 2$. For $k > n$ this leading anomaly coefficient vanishes.

One can understand the differences between the cases $k \leqslant n$ and $k > n$ that lead to this restriction in several ways. For $2 \leqslant k \leqslant n$ we have $\Pi_{2k-2}(\mathrm{SU}(n)) = \boldsymbol{Z}$ and these groups correspond to the non-trivial Casimirs of SU(n). In the unstable range $k > n$ things rapidly become more complicated. We present the example of SU(3) to illustrate how one may calculate beyond this range if required and to illustate why such complications arise. The major aim however is to present the general tables $A_{(k)}(\lambda)$. We limit ourselves to those irreducible SU(n) representations whose corresponding Young diagrams have fewer than seven boxes. This imposes no restrictions on those representations commonly considered by model builders. In an appendix we give the algorithm employed in constructing these tables so further representations may be similarly calculated for.

Calculating the leading anomaly coefficients is purely an algebraic question and there are now several methods for their construction. These are reviewed, for example, in Braden (1988b). General expressions for some $A_{(k)}(\lambda)$ have been derived by many authors for certain λ and values of k. When λ is a totally antisymmetric tensor representation of SU(n) then there are several equivalent expressions that give $A_{(k)}(\lambda)$ for $k \leqslant n$ (Frampton and Kephart 1983a, b, Okubo and Patera 1983, 1984, Braden 1988b). Again when λ has a diagram of four or fewer boxes or two or less rows or columns results also exist (Okubo and Patera 1983). Also for Young diagrams of six and fewer boxes Tosa and Okubo (1988)† give separate expressions for $A_{(3)}(\lambda)$, $A_{(5)}(\lambda)$ and $A_{(7)}(\lambda)$ as a polynomial in n which are reproduced by our general polynomials. Our calculational method was outlined by Braden (1988b) and provides an easily implemented algorithm for constructing the polynomials $A_{(k)}(\lambda)$ simultaneously for all $k \leqslant n$‡. This method is easily described. Any irreducible representation of a Lie algebra g has weights which decompose into orbits of the Weyl group of g. For each such orbit we can construct a simpler polynomial $a_{(k)}$ from which we reconstruct $A_{(k)}$ by summing over the orbits. This will be illustated in the next section.

2. Calculating $A_{(k)}(\lambda)$ for $2 \leqslant k \leqslant n$

We will now describe how to calculate the anomaly coefficients and give some examples of such a calculation. The results of this section are given in the tables 1-3.

Let us recall there is a one-to-one correspondence between the finite-dimensional unitary irreducible representations of SU(n) and the dominant weights of its weight lattice Λ^+. Also with each dominant weight we may associate a Young diagram $(l) = (l_1, l_2, \ldots, l_N)$ with N rows of length $l_i \neq 0$ (for $U(n)$ $N \leqslant n$ while for SU(n) the

† Note that in these authors' notation $A_{(k)}(\lambda) \equiv Q_{(k)}(\lambda)$.

‡ We take this opportunity to correct some typographical errors in this reference.

Table 1. The leading anomaly coefficients $A_{(k)}(\lambda)$ for SU(n) for Young diagrams λ of up to five boxes.

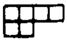
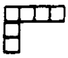
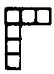
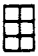
λ	$A_{(k)}(\lambda)$
	$n+2^{k-1}$
	$n-2^{k-1}$
	$\frac{1}{2}n^2+\frac{1}{2}n(2^k+1)+3^{k-1}$
	n^2-3^{k-1}
	$\frac{1}{2}n^2-\frac{1}{2}n(2^k+1)+3^{k-1}$
	$(2n^3+3n^2(2^k+2)+n(4.3^k+3.2^k+4)+3.4^k)/12$
	$(2n^3+n^2(2^k+2)-n2^k-4^k)/4$
	$(2n^3-n(2.3^k-3.2^k+2))/6$
	$(2n^3-n^2(2^k+2)-n2^k+4^k)/4$
	$(2n^3-3n^2(2^k+2)+n(4.3^k+3.2^k+4)-3.4^k)/12$
	$(5n^4+10n^3(2^k+3)+5n^2(4.3^k+6.2^k+11)+10n(3.4^k+2.3^k+2.2^k+3)$ $+24.5^k)/5!$
	$(5n^4+5n^3(2^k+3)+5n^2(3^k+2)-5n(3^k+2^k)-6.5^k)/30$
	$(5n^4+2n^3(2^k+3)-n^2(4.3^k-6.2^k+5)-2n(3.4^k-2.3^k-2.2^k+3))/24$
	$(5n^4-5n^2(2.2^k+1)+4.5^k)/20$
	$(5n^4-2n^3(2^k+3)-n^2(4.3^k-6.2^k+5)+2n(3.4^k-2.3^k-2.2^k+3))/24$
	$(5n^4-5n^3(2^k+3)+5n^2(3^k+2)+5n(3^k+2^k)-6.5^k)/30$
	$(5n^4-10n^3(2^k+3)+5n^2(4.3^k+6.2^k+11)-10n(3.4^k+2.3^k+2.2^k+3)$ $+24.5^k)/5!$

inequality is strict). Explicitly if λ_i is a fundamental weight it has Young diagrams given by $l_j=1$ for $j\leq i$ and zero otherwise. Let us denote by l_λ the Young diagram associated with a weight λ and $V(\lambda)$ the set of weights of an irreducible representation with highest weight λ. We set $m_\lambda(\mu)$ the multiplicity of the weight μ in $V(\lambda)$; these may be calculated by several recursions and are extensively tabulated (Bremner *et al* 1985).

With this notation we construct $A_{(k)}(\lambda)$ out of simpler functions $a_{(k)}(\mu)$ in the manner already described:

$$A_{(k)}(\lambda)=\sum_{\mu\in V(\lambda)\cap\Lambda+} m_\lambda(\mu)a_{(k)}(\mu). \tag{2.1}$$

Table 2. The leading anomaly coefficients $A_{(k)}(\lambda)$ for SU(n) for Young diagrams λ of six boxes.

λ	$A_{(k)}(\lambda)$
	$(6n^5+15n^4(2^k+4)+10n^3(4.3^k+9.2^k+21)+15n^2(6.4^k+8.3^k+11.2^k+20)$ $+2n(72.5^k+45.4^k+40.3^k+45.2^k+72)+120.6^k)/6!$
	$(6n^5+9n^4(2^k+4)+2n^3(8.3^k+9.2^k+33)+9n^2(2.4^k-2^k+4)-2n(9.4^k+8.3^k+9.2^k)$ $-24.6^k)/144$
	$(6n^5+5n^4(2^k+4)+10n^3(2^k+1)-5n^2(2.4^k-3.2^k+4)-2n(8.5^k-5.4^k-5.2^k+8))/80$
	$(6n^5+3n^4(2^k+4)+2n^3(2.3^k-9.2^k-3)-3n^2(4.3^k+7.2^k+4)+8n3^k+12.6^k)/72$
	$(6n^5+3n^4(2^k+4)-2n^3(4.3^k-9.2^k+3)-3n^2(6.4^k-8.3^k+2^k+4)$ $-2n(9.4^k-16.3^k+9.2^k))/144$
	$(6n^5-5n^3(3^k+3)+n(9.5^k-10.3^k+9))/45$
	$(6n^5-3n^4(2^k+4)+2n^3(2.3^k-9.2^k-3)+3n^2(4.3^k+7.2^k+4)+8n3^k-12.6^k)/72$
	$(6n^5-3n^4(2^k+4)-2n^3(4.3^k-9.2^k+3)+3n^2(6.4^k-8.3^k+2^k+4)$ $-2n(9.4^k-16.3^k+9.2^k))/144$
	$(6n^5-5n^4(2^k+4)+10n^3(2^k+1)+5n^2(2.4^k-3.2^k+4)-2n(8.5^k-5.4^k-5.2^k+8))/80$
	$(6n^5-9n^4(2^k+4)+2n^3(8.3^k+9.2^k+33)-9n^2(2.4^k-2^k+4)$ $-2n(9.4^k+8.3^k+9.2^k)+24.6^k)/144$
	$(6n^5-15n^4(2^k+4)+10n^3(4.3^k+9.2^k+21)-15n^2(6.4^k+8.3^k+11.2^k+20)$ $+2n(72.5^k+45.4^k+40.3^k+45.2^k+72)-120.6^k)/6!$

In an appropriate basis $m_\lambda(\mu)$ is an upper diagonal matrix with integer entries. Finally we need (for $(l_\mu)=(l_1, l_2, \ldots, l_N)$)

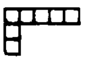
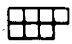
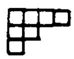
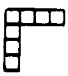
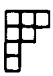
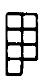
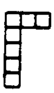
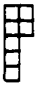
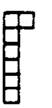
$$a_{(k)}(\mu)=\kappa_\mu \sum_{p=1}^{N}(-1)^{p+1}\frac{(p-1)!\,n!}{(n-N+p)!}\sum(x_1+x_2+\ldots+x_p)^k[l_\mu]. \tag{2.2}$$

The notation is as follows. The final sum means we sum over distinct choices of p elements x_i from $(x_1, \ldots, x_N)$ and then evaluate this by substituting the corresponding l_i. Finally

$$\kappa_\mu=\frac{|\text{Orbit }\mu|(n-N)!}{n!}. \tag{2.3}$$

If we write $(l_\mu)=(p_1^{\pi_1}, p_2^{\pi_2}, \ldots, p_s^{\pi_s})$ where $p_1>p_2>\ldots>p_s>0$ then $\kappa_\mu=1/\pi_1!\pi_2!\ldots\pi_s!$. Expression (2.2) may be obtained from formulae given in Braden (1988b); this is done in appendix A while appendix B describes a simple recursive algorithm for its implementation. Tables 1-3 summarise the results of calculating

Table 3. The leading anomaly coefficients $A_{(k)}(\lambda)$ for SU(n) for Young diagrams λ of seven boxes.

λ	$A_{(k)}(\lambda)$
	$(7n^6+21n^5(2^k+5)+35n^4(2.3^k+6.2^k+17)+105n^3(2.4^k+4.3^k+7.2^k+15)$ $+14n^2(36.5^k+45.4^k+55.3^k+75.2^k+137)$ $+84n(10.6^k+6.5^k+5.4^k+5.3^k+6.2^k+10)+720.7^k)/7!$
	$(7n^6+14n^5(2^k+5)+35n^4(3^k+2.2^k+7)+70n^3(4^k+3^k+2^k+5)$ $+7n^2(12.5^k-5.3^k-10.2^k+24)-14n(6.5^k+5.4^k+5.3^k+6.2^k)-120.7^k)/840$
	$(7n^6+9n^5(2^k+5)+5n^4(2.3^k+6.2^k+17)+15n^2(3.2^k+1)$ $-2n^2(18.5^k+5.3^k-30.2^k+46)-12n(5.6^k-3.5^k-3.2^k+5))/360$
	$(7n^6+7n^5(2^k+5)+7n^4(2.3^k-2.2^k+5)+7n^3(2.4^k-4.3^k-13.2^k-5)$ $-14n^2(3.4^k+3^k+5.2^k+3)+28n(4^k+3^k)+48.7^k)/336$
	$(7n^6+6n^5(2^k+5)-5n^4(3^k-6.2^k-5)-30n^3(4^k-3^k-2^k+1)$ $-n^2(36.5^k-65.3^k+30.2^k+32)-6n(6.5^k-5.4^k-5.3^k+6.2^k))/360$
	$(7n^6+3n^5(2^k+5)-n^4(2.3^k+6.2^k+17)-3n^3(2.4^k+4.3^k+2^k+13)$ $+2n^2(9.4^k-11.3^k+3.2^k+5)+12n(2.6^k-4^k-3^k+2))/144$
	$(7n^6+n^5(2^k+5)-5n^4(2.3^k-2.2^k+5)-5n^3(2.4^k-4.3^k+5.2^k+1)$ $+2n^2(12.5^k-15.4^k+5.3^k-5.2^k+9)+4n(6.5^k-5.4^k-5.3^k+6.2^k))/240$
	$(7n^6+7n^4(3^k-6.2^k-5)+7n^2(11.3^k+6.2^k+4)-36.7^k)/252$
	$(7n^6-n^5(2^k+5)-5n^4(2.3^k-2.2^k+5)+5n^3(2.4^k-4.3^k+5.2^k+1)$ $+2n^2(12.5^k-15.4^k+5.3^k-5.2^k+9)-4n(6.5^k-5.4^k-5.3^k+6.2^k))/240$
	$(7n^6-3n^5(2^k+5)-n^4(2.3^k+6.2^k+17)+3n^3(2.4^k+4.3^k+2^k+13)$ $+2n^2(9.4^k-11.3^k+3.2^k+5)-12n(2.6^k-4^k-3^k+2))/144$
	$(7n^6-6n^5(2^k+5)-5n^4(3^k-6.2^k-5)+30n^3(4^k-3^k-2^k+1)$ $-n^2(36.5^k-65.3^k+30.2^k+32)+6n(6.5^k-5.4^k-5.3^k+6.2^k))/360$
	$(7n^6-7n^5(2^k+5)+7n^4(2.3^k-2.2^k+5)-7n^3(2.4^k-4.3^k-13.2^k-5)$ $-14n^2(3.4^k+3^k+5.2^k+3)-28n(4^k+3^k)+48.7^k)/336$
	$(7n^6-9n^5(2^k+5)+5n^4(2.3^k+6.2^k+17)-15n^3(3.2^k+1)-2n^2(18.5^k+5.3^k-30.2^k+46)$ $+12n(5.6^k-3.5^k-3.2^k+5))/360$
	$(7n^6-14n^5(2^k+5)+35n^4(3^k+2.2^k+7)-70n^3(4^k+3^k+2^k+5)$ $+7n^2(12.5^k-5.3^k-10.2^k+24)+14n(6.5^k+5.4^k+5.3^k+6.2^k)-120.7^k)/840$
	$(7n^6-21n^5(2^k+5)+35n^4(2.3^k+6.2^k+17)-105n^3(2.4^k+4.3^k+7.2^k+15)$ $+14n^2(36.5^k+45.4^k+55.3^k+75.2^k+137)$ $-84n(10.6^k+6.5^k+5.4^k+5.3^k+6.2^k+10)+720.7^k)/7!$

$A_{(k)}(\lambda)$ using this method. We conclude this section by two examples of such a calculation.

First we note for $N=1, 2, 3$

$$
\begin{aligned}
&\text{(i)} && a_{(k)}(l_1)=l_1^k \\
&\text{(ii)} && a_{(k)}(l_1, l_2)=\kappa_{(l_1,l_2)}\{n(l_1^k+l_2^k)-(l_1+l_2)^k\} \qquad (2.4)\\
&\text{(iii)} && a_{(k)}(l_1, l_2, l_3)=\kappa_{(l_1,l_2,l_3)}\left\{\begin{matrix} n^2(l_1^k+l_2^k+l_3^k)+2(l_1+l_2+l_3)^k \\ -n((l_1+l_2)^k+(l_2+l_3)^k+(l_3+l_1)^k+l_1^k+l_2^k+l_3^k)\end{matrix}\right\}.
\end{aligned}
$$

Example 1. For representations with Young diagrams with two boxes equation (2.1) yields

$$
\begin{pmatrix} A_{(k)}(\square\square) \\ A_{(k)}(\substack{\square\\\square}) \end{pmatrix}=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_{(k)}(\square\square) \\ a_{(k)}(\substack{\square\\\square}) \end{pmatrix} \qquad (2.5)
$$

and substitution of the first two equations of (2.4) with $\kappa_{(a,a)}=\frac{1}{2}$ gives

$$
\begin{aligned}
\begin{pmatrix} A_{(k)}(\square\square) \\ A_{(k)}(\substack{\square\\\square}) \end{pmatrix}&=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2^k \\ n-2^{k-1} \end{pmatrix} \\
&=\begin{pmatrix} n+2^{k-1} \\ n-2^{k-1} \end{pmatrix}.
\end{aligned} \qquad (2.5)
$$

This is well known for general k.

Example 2. We repeat the calculation this time for Young diagrams of three boxes noting now that $\kappa_{a,a,b}=\kappa_{a,b,b}=\frac{1}{2}(b\neq a)$ and $\kappa_{a,a,a}=\frac{1}{6}$. Then

$$
\begin{aligned}
\begin{pmatrix} A_{(k)}(\square\square\square) \\ A_{(k)}(\substack{\square\square\\\square}) \\ A_{(k)}(\substack{\square\\\square\\\square}) \end{pmatrix}&=\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_{(k)}(\square\square\square) \\ a_{(k)}(\substack{\square\square\\\square}) \\ a_{(k)}(\substack{\square\\\square\\\square}) \end{pmatrix} \\
&=\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3^k \\ n(2^k+1)-3^k \\ n^2/2-n(2^k+1)/2+3^{k-1} \end{pmatrix} \\
&=\begin{pmatrix} n(n+1)/2+n2^{k-1}+3^{k-1} \\ n^2-3^{k-1} \\ n(n-1)/2-n2^{k-1}+3^{k-1} \end{pmatrix}.
\end{aligned}
$$

This reproduces the results of Okubo and Patera (1983).

3. The unstable range: an example

Although not relevant for the calculation of perturbative anomalies this section is included to illustrate what is happening in the unstable range $k>n$. Such calculations are however germane for global anomalies (Braden 1988a, b).

The general algebraic problem we are discussing when writing (1.1) is the decomposition of a group invariant polynomial into a given basis of invariant polynomials, here traces and their powers of a given fundamental representation. For any given simple Lie group this basis is finite dimensional and a possible basis is $\{\mathrm{Tr}_{V(\lambda_f)}F^{\nu_i}\}$ where ν_i are the exponents of the Lie group. In the case of SU(n) with λ_f the n-dimensional representation, we have $\nu_i \in \{2, 3, \ldots, n\}$ and the algebraic question is one of expressing a symmetric polynomial in terms of the symmetric power functions $s_k = \Sigma_{i=1}^{n} x_i^k$. The unstable range corresponds to expressing $\mathrm{Tr}_{V(\lambda_f)}\, F^k$ in terms of our basis when k is larger than the maximum exponent of the group. Clearly in this case there can be no leading anomaly coefficient as defined.

To be concrete let us take the example of SU(3) which has fundamental weights λ_1 and λ_2 which we associate with 3 and $\bar{3}$ respectively. The Young diagram of an SU(3) irreducible representation with highest weight $n_1\lambda_1 + n_2\lambda_2$ and $l_\mu = (n_1 + n_2, n_2)$. The algebraic problem we have reduces to finding the coefficients $A_{(m,n)}(\lambda)$ in the expression

$$\sum_{\mu \in V(\lambda)} \mu^k = \sum_{2m+3n=k} A_{(m,n)}(\lambda)\left(\sum_{\mu_1 \in V(\lambda_1)} \mu_1^2\right)^m \left(\sum_{\mu_2 \in V(\lambda_1)} \mu_2^3\right)^n. \tag{3.1}$$

Our general approach outlined in the previous section has been to rewrite this as

$$\sum_{\mu \in V(\lambda)} \mu^k = \sum_{\mu \in V(\lambda) \cap \Lambda^+} m_\lambda(\mu) f_{(k)}(\mu) \tag{3.2}$$

where

$$f_{(k)}(\mu) = \frac{|\mathrm{Orbit}(\mu)|}{6} \sum_{\sigma \in S_3} \sigma\mu^k$$

$$= \sum_{2m+3n=k} a_{(m,n)}(\mu)\left(\sum_{\mu_1 \in V(\lambda_1)} \mu_1^2\right)^m \left(\sum_{\mu_2 \in V(\lambda_1)} \mu_2^3\right)^n \tag{3.3}$$

whence

$$A_{(m,n)}(\lambda) = \sum_{\mu \in V(\lambda) \cap \Lambda^+} m_\lambda(\mu) a_{(m,n)}(\mu). \tag{3.4}$$

In calculations with SU(n) it is natural to express the weight space as the hyperplane of $\boldsymbol{R}^n$ orthogonal to $\zeta = (1/n)(x_1 + x_2 + \ldots + x_n)$. On this subspace we have $\{x_1, \ldots, x_n\}$ being the set of weights of $V(\lambda_1)$ and the Weyl group acts as the symmetric group S_n. When this is the case we have $\Sigma_{\mu \in V(\lambda_1)}\, \mu^k = \Sigma_{i=1}^{n} x_i^k = s_k$ and because we restrict attention to the hyperplane perpendicular to ζ then $s_1 = 0$ on this space.

For the particular case of su(3) and $\mu = n_1\lambda_1 + n_2\lambda_2$ this means we wish to calculate

$$f_{(k)}(\mu) = \frac{|\mathrm{Orbit}(\mu)|}{6} \sum_{\sigma \in S_3} \sigma(l_1 x_1 + l_2 x_2)^k$$

$$= \sum_{2m+3n=k} a_{(m,n)}(\mu) s_2^m s_3^n.$$

To calculate $a_{(m,n)}$ we may proceed as follows. First

$$f_{(k)}(\mu) = \frac{|\mathrm{Orbit}(\mu)|}{6} \sum_{p=0}^{k} \binom{k}{p} l_1^p l_2^{k-p} \sum_{\sigma \in S_3} \sigma(x_1^p x_2^{k-p}).$$

Upon noting

$$(x_1^p + x_2^p + x_3^p)(x_1^{k-p} + x_2^{k-p} + x_3^{k-p}) = \sum_{\sigma \in S_3} \sigma(x_1^p x_2^{k-p}) + s_k$$

then

$$f_{(k)}(\mu)=\kappa_\mu\sum_{p=0}^{k}\binom{k}{p}l_1^p l_2^{k-p}[s_p s_{k-p}-s_k] \tag{3.5}$$

where $\kappa_\mu=1/2$ when $l_1=l_2$ or $l_2=0$, and $\kappa_\mu=1$ otherwise. (Note $s_0=3$ and $s_i=0$, $i<0$.)

It is at this stage of the calculation we can see the difference that arises when $k>3$. In this range s_k is a function of s_2 and s_3 and we must make this explicit. In the stable range we can just read off the leading anomaly coefficient directly and this leads to (2.2). For su(n) it is straightforward to recursively define the $s_k (k>n)$ in terms of the lower s_k. To do this introduce the elementary symmetric functions e_k by $\Pi_{i\geq 1}(1+x_i t)=\Sigma_{k\geq 0}t^k e_k$. Then we have Newton's formula

$$s_k=e_1 s_{k-1}-e_2 s_{k-2}+e_3 s_{k-3}-\ldots+(-1)^{n-1}e_n s_{k-n}$$

where $s_i=0 (i<0)$ and $s_0=n$. By expressing the e_i in terms of the power symmetric functions we have the desired recurrence.

For the su(3) example we find $s_1=e_1=0$, $s_2=-2e_2$, $s_3=3e_3$ and $s_k=\frac{1}{3}s_3 s_{k-3}+\frac{1}{2}s_2 s_{k-2}$ for $k\geq 4$. In particular $s_4=\frac{1}{2}s_2^2$, $s_5=\frac{5}{6}s_2 s_3$ and $s_6=\frac{1}{3}s_3^2+\frac{1}{4}s_2^3$.

Using these results we find for example that

$$f_{(4)}(\mu)=\kappa_\mu(n_1^4+2n_1^3 n_2+3n_1^2 n_2^2+2n_1 n_2^3+n_2^4)s_2^2=a_{(2,0)}(\mu)s_2^2. \tag{3.6}$$

Now for su(3) the multiplicities $m_\lambda(\mu)$ have a particularly simple structure (Antoine and Speiser 1964). If $\lambda=m\lambda_1+n\lambda_2\equiv(m,n)$ and taking $m\geq n$ with no loss of generality then

(i) $m_\lambda(\mu)=k+1, 0\leq k\leq n$ for weights $\mu=(n_1,n_2)$ of the form: $(m-k,\ n-k)$; $(m-k+j,\ n-k-2j)$ and $j:1,\ldots,[\frac{n-k}{2}]$; $(m-k-2j,\ n-k+j)$ and $j:1,\ldots[\frac{1}{2}(m-k)]$.

(ii) $m_\lambda(\mu)=n+1$ for weights of the form: $(m-n-3r-2s,\ s)$ where $r:1,\ldots[\frac{1}{3}(m-n)]$ and $s:0,\ldots[\frac{1}{2}(m-n-3r)]$.

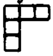
Utilising this information we may perform the summation (3.4) to obtain for $k:2,\ldots 5$,

$$\begin{aligned}
A_{(1,0)}(m,n)&=\tfrac{1}{24}(m+1)(n+1)(m+n+2)[m^2+n^2+mn+3m+3n]\\
A_{(0,1)}(m,n)&=\tfrac{1}{60}(m-n)(m+1)(n+1)(m+n+2)[2m^2+2n^2+5mn+9m+9n+9]\\
A_{(2,0)}(m,n)&=\tfrac{1}{10}A_{(1,0)}(m,n)[2m^2+2n^2+2mn+6m+6n-3]\\
A_{(1,1)}(m,n)&=\tfrac{5}{42}A_{(0,1)}(m,n)[4m^2+4n^2+4mn+12m+12n-9].
\end{aligned} \tag{3.7}$$

The first three of these expressions may be obtained from the work of Okubo (1982) and Okubo and Patera (1983) upon noting the dimension of $V(\lambda)$ is given by $\dim(m,n)=\frac{1}{2}(m+1)(n+1)(m+n+2)$.

4. Discussion

Because tables 1-3 are intended for those wishing to know simply the anomaly content of su(n) gauge theories we will give some examples of their application. Before doing this, however, we note several non-trivial checks of these tables. If $\bar{\lambda}$ denotes the complex conjugate representation to λ and λ^{T} the representation with diagram conjugate (i.e. interchanging rows and columns) to λ then $A_{(k)}(\bar{\lambda})=(-1)^k A_{(k)}(\lambda)$ and the formula for $A_{(k)}(\lambda^{\mathrm{T}})$ may be obtained from that of $A_{(k)}(\lambda)$ by substituting $n\to -n$ up to an overall sign $(-1)^{|\lambda|-1}$ where $|\lambda|$ is the number of boxes making up λ. These observations were described in Okubo and Patera (1983). Together these provide powerful checks on the polynomials $A_{(k)}(\lambda)$.

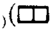
Take for example $\lambda_1 =$  and $\lambda_2 =$ . These diagrams are symmetric about their diagonal whence λ_1 is an even polynomial in n and λ_2 an odd polynomial. Further, for $n=3$ we have $\lambda_2 = \bar{\lambda}_1$. Thus for $2 \leq k \leq 3$ and $n=3$ we must have $A_{(k)}(\bar{\lambda}_1) = (-1)^k A_{(k)}(\lambda_2)$, that is $8 - 3^k + 3.2^{k-1} = (-1)^k(18 + 5^{k-1} - 9.2^{k-1})$. This illustrates some of the non-trivial relations $A_{(k)}(\lambda)$ must satisfy.

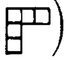
To illustrate the use of our tables we consider the example of an su(5) theory in eight dimensions, hence we are interested in $A_{(5)}(\lambda)$. This example has been chosen simply because it has been discussed by several authors and we may compare various results. We will follow the labelling conventions of McKay and Patera (1981) which also includes the su(5) $\supset$ su(4) reduction

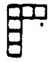
 $\supset$  +  +  + .

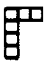
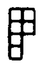
40 20 10 6 4

Comparing this reduction with that given in Elitzur and Nair (1984) we see their conventions interchange the 40 and $\overline{40}$ of our adopted labelling and similarly the 35 and $\overline{35}$. Our tables yield

$A_{(5)}($$) = -11$ $A_{(5)}($$) = 21$ $A_{(5)}($$) = -56$

$A_{(5)}($$) = 176$ $A_{(5)}\left(\right.$$\left.\right) = 89$ $A_{(5)}\left(\right.$$\left.\right) = 66.$

Together with $A_{(5)}(\ ) = 1$ these reproduce the results of these authors†. Comparison with the table of Holman and Kephart (1986) however reveals some discrepancies, the simplest of which to check is for the 70-dimensional representation . We have

 $\otimes$  =  +  + 

dim	$5 \otimes 24$	=	70	+	$\overline{45}$	+	5
$A_{(5)}$	$5 \times 0 + 24 \times 1$	=	89	+	-66	+	1

where we have used the property of the leading anomaly coefficients (Frampton and Kephart 1983a, Okubo and Patera 1983)

$$A_{(k)}(\lambda \otimes \mu) = \dim \lambda A_{(k)}(\mu) + \dim \mu A_{(k)}(\lambda) = \sum_{\rho \in \lambda \otimes \mu} A_{(k)}(\rho)$$

this direct calculation confirming the tables.

Hopefully this example illustrates some of the utility of our tables and will provide a helpful tool to those wishing to simply calculate anomaly cancellation in non-vector-like theories. A computer file with these tables may be e-mailed to any reader who desires them.

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† There is an overall sign difference from their final example possibly arising from labelling conventions.

Appendix A

Here we will derive expression (2.2) for $a_{(k)}(\mu)$. We begin with equation (3.45) of Braden (1988b),

$$a_{(k)}(\mu) = |\text{Orbit}\,\mu| \sum_{j=1}^{k} (-1)^{j-1} \frac{(j-1)!(n-j)!}{n!} F_N(k,j)(\mu) \tag{A1}$$

where $F_N(k,j)$ is a polynomial defined in the reference cited with the property $F_N(k,j)=0$ when $j>k$. Because we are working in the stable range, $k \leq n$, the vanishing property of $F_N(k,j)$ means we can replace the summation in (A1) by $\Sigma_{j=1}^{n}$. In these formulae N is the number of rows of l_μ, the Young diagram associated with μ. We have $|\text{Orbit}(\mu)| = \kappa_\mu n!/(n-N)!$ where κ_μ was defined by (2.3). Now using expression (3.47) of Braden (1988b) for $F_N(k,j)$ these substitutions yield

$$a_{(k)}(\mu) = \kappa_\mu \sum_{j=1}^{n} (-1)^{j-1} \frac{(j-1)!(n-j)!}{(n-N)!} \sum_{r=0}^{j-1} (-1)^r \binom{N-j+r}{r} \sum (x_1+\ldots+x_{j-r})^k(\mu)$$

where the final summation is over distinct choices of $j-r$ variables from $\{x_1, \ldots, x_N\}$. Letting $p=j-r$ and rearranging some of the factorials then gives

$$a_{(k)}(\mu) = \kappa_\mu \sum_{p=1} (-1)^{p+1}(p-1)!(N-p)! \sum_{j=1}^{n} \binom{j-1}{p-1}\binom{n-j}{n-N} \sum (x_1+\ldots+x_{j-r})^k(\mu)$$

and the result follows from

$$\sum_{j=1}^{n} \binom{j-1}{p-1}\binom{n-j}{n-N} = \binom{n}{N-p}.$$

Appendix B

This contains an algorithm for generating the polynomials $a_{(k)}(\lambda)$ using a symbolic manipulation language such as REDUCE. The only term in (2.2) which is non-trivial to translate is

$$\sum (x_1+\ldots+x_p)^k[l_\mu]. \tag{B1}$$

This can be described as a sum over all distinct ways of selecting p components of l_μ. By distinct we mean that l_i is distinct from l_j when the labels i, j are different, even if the values of the components themselves are equal. So choosing a particular component is equivalent to choosing its label and we can write the ith choice as $x_i = l_{s_i}$ for $1 \leq s_i \leq N$. The labels $(s_1, \ldots, s_p)$ define the components of l_μ to be chosen and form an unordered set as different orderings do not correspond to distinct selections. We must find an exhaustive method of selecting these labels, and we will develop a recursive algorithm to do so. The problem is equivalent to finding all the distinct patterns when placing p crosses in a row of N boxes. Let the boxes be numbered 1 to N from left to right. We suppose $j-1$ crosses have already been placed and that in previous selections patterns with these crosses to the left of their present positions have all been considered. Then the next cross must be placed in a box to the right of the previous cross or else a pattern will be repeated. Now s_j is the number of the box in which the jth cross is to be placed and so we have

$$s_{j-1}+1 \leq s_j \leq N-p+j. \tag{B2}$$

The upper limit is due to the restriction that there must be enough room to the right of the jth cross for the remaining crosses. The first cross can be placed as far to the left as possible, but the upper limit still applies so $1 \leq s_1 \leq N-p+1$.

Inequality (B2) defines a 'node' in a tree-like structure with each value of s_j causing a branch in that tree. At the end of each branch is another node, and so on. The number of branches at each node is $N-p+j-s_{j-1}$; hence the tree grows very quickly initially, with less branches later on. This can be coded very succinctly into a procedural function **sum** which returns the sum (B1):

$$
\begin{aligned}
&\textbf{define function sum}\ (p, N, s_{\min}, j, X)\\
&\qquad \textbf{if}\ j = p+1\ \textbf{then return}\ X^k\\
&\qquad \textbf{else return}\ \sum_{s=s_{\min}}^{N-p+j} \textbf{sum}\ (p, N, s+1, j+1, X+l_s)\\
&\qquad \textbf{end.}
\end{aligned}
\tag{B3}
$$

This would be called by **sum** $(p, N, 1, 1, 0)$. When **sum** is called with a given j the value $s_{\min}$ is the lower limit of s_j as defined by the inequality (B2). In the third line (which is the node) the sum forms all the branches at that node. Note that the expression $x_1 + \ldots + x_p$ is built up in X. The final node occurs in the second line where the expression $(x_1 + \ldots + x_p)^k$ is returned to the previous node. And finally calling the function from outside with s and j set to one, will start with s_1 running from 1 to $N-p+1$.

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